II. A Figure of Solid Analytic Geometry.

By John W. Bradshaw, University of Michigan.

There may well be a difference of opinion as to the particular system of axonometry that should be used in the figures of solid analytic geometry. Some would prefer drawings in perspective, as giving the best likeness and therefore furnishing the greatest assistance to the student's geometrical imagination; others would choose the simplest form of oblique parallel projection, that in which the picture-plane is assumed parallel to one of the coördinate planes, as lending itself most readily to reproduction by the student. The former type of figure is represented chiefly in the resort to photography of models,—Is there here a confession of inadequacy on the part of the draftsman?—the latter type is that commonly found in the texts. Perhaps a judicious use of both types with a comparison of the two would render the greatest service to the student.

Whatever attitude be taken on this question, however, it seems hardly necessary to contend that when a decision is once reached each figure should be consistently and accurately drawn according to the system chosen. Nor is much argument needed to show that definiteness is desirable, such definiteness as is attained by marking on the pictures of the coördinate axes three segments to represent three equal segments, or units, on the axes in space. Unless some such indication is given, one has no means of determining the point, or the direction, from which the figure should be viewed, a matter of the utmost importance if it is to produce the best possible impression. The drawings in most English and American texts with which I am acquainted are little short of disgraceful in their careless disregard of truth.

Let me call attention to the defects of just one simple figure as it appears in many books, the figure that accompanies the derivation of the equation of a plane in the normal form. Figure 1, but for the dotted lines and the lettering, is copied from a well-known text. To facilitate the description and distinguish relations in space from those of the plane figure, I shall designate points in space by Roman capitals and their pictures by the same letters starred.

In Figure 1, then, O^*X^*, O^*Y^*, and O^*Z^* represent three mutually perpendicular axes in space, OX, OY, and OZ. A plane η cuts these axes in U, V, W, represented by U^*, V^*, W^*. A perpendicular is let fall upon this plane from the origin, N is the foot of this perpendicular, O^*N^* is its picture. P is any point of the plane, reached from the origin by the broken line $OMLP$, whose parts OM, ML, LP give the coördinates of P. Since O^*X^* and O^*Z^* are perpendicular to each other it seems a reasonable assumption that the author had in mind a picture-plane parallel to the plane XOZ, though this cannot be positively asserted since he does not indicate the units. The first criticism of the figure does not depend on this assumption, the second does.

The point P, as end of the broken line $OMLP$, is not a point of the plane η. The dotted lines O^*K^* and K^*W^* are drawn to show this. The parallels OZ and LP determine a plane which cuts the plane XOY in OK and the plane η

in KW. If P lies in η it must lie in KW; so that Q^*, and not P^*, should be taken to represent a point of the plane. To be sure P^* is the picture of some point of the plane but not the point whose coördinates are indicated.

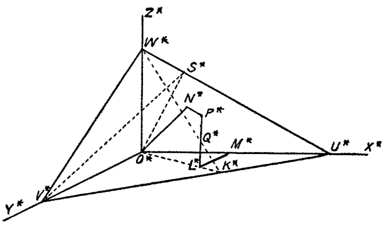

FIG. 1.

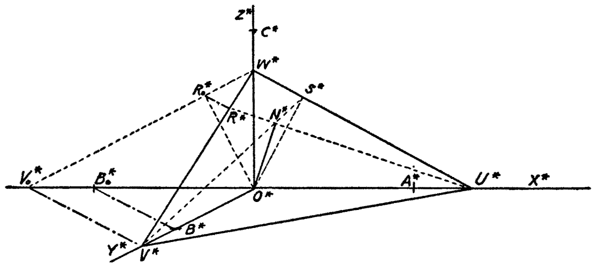

FIG. 2.

Again, N is an impossible position for the foot of the normal. For the plane ν of ON and OY is perpendicular both to XOZ and to η, and hence to their intersection UW. It cuts XOZ in a line OS perpendicular to UW. This must be represented by O^*S^* perpendicular to U^*W^*, since in this plane perpendiculars are represented by perpendiculars. Further, ν cuts η in VS and N must lie on this line, N^* on V^*S^*. The exact position of N^* cannot be given unless the unit on O^*Y^* be marked.

In Figure 2, O^*A^*, O^*B^*, O^*C^* represent equal segments on the axes in space, and the position of N is determined by rabatting the plane YOZ about the Z-axis till it coincides with XOZ. B^* thereby falls at B_0^*, such that $O^*B_0^*=O^*A^*$. The construction proceeds by drawing $V^*V_0^* \| B^*B_0^*$, $O^*R_0^* \perp V_0^*W^*$, $R_0^*R^* \| B_0^*B^*$. Then N^* is given as the intersection of U^*R^* and V^*S^*. We might instead have rabatted XOY about OX into coincidence with XOZ. The two constructions combined yield a useful check on the accuracy of the drawing.

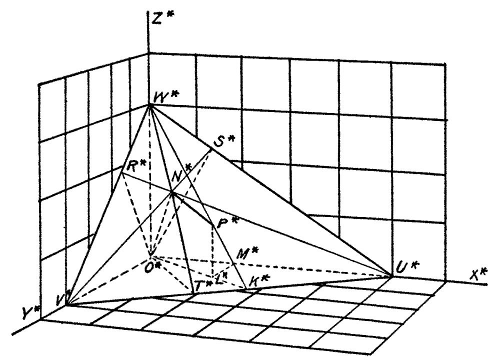

FIG. 3

As an illustration of what may be done with perspective Fig: 3 is offered. Would not many a student for whom Fig. 1 presents difficulties find Fig. 3 easily readable?

Has perhaps the author of the Solid Analytics been tricked into a careless and unwarranted sense of security by the generality of the theorem of Pohlke, that fundamental theorem of axonometry which asserts that any three segments of a plane, O^*X^*, O^*Y^*, O^*Z^* emanating from a point, so long as not more than three of the four points lie in a line, may be considered as the parallel projection of three equal, mutually perpendicular segments of space? Their lengths may

be taken at pleasure and the angles that they form with each other are entirely arbitrary. A superficial view of this theorem might lead one to think that the figures of axonometry may be drawn haphazard and that they will then represent the space figures in mind from some point of vision. On closer acquaintance, however, the theorem is seen to be associated with law, rather than license. It relates to setting up in the plane figure a standard of measurement which makes for definiteness.

RECENT PUBLICATIONS.

REVIEWS.

Girolamo Saccheri's Euclides Vindicatus. Edited and translated by GEORGE BRUCE HALSTED. Chicago and London, Open Court, 1920. 8vo. 30 + 246 pp. Price $2.00.

The Jesuit Girolamo Saccheri was born in 1667 and first taught in a college of his order in Milan where Tommaso Ceva, a brother of the Giovanni Ceva whose triangle theorem is well known, was teacher of mathematics. The influence of these brothers is apparent in the first two mathematical works which Saccheri published: (1) *Quæsita geometrica a Comite Rugerio de Vigintimilliis omnibus proposita, ab Hieronymo Saccherio Genuensi Societatis Jesu soluta.* Mediolani, 1693 (37 pages); another edition Parma, 1694 (dealing mainly with the solution of six problems in conic sections). (2) *Neostatica auctore Hieronymo Saccherio e Societate Jesu Excellentissimo senatui Mediolanensi dicata.* Mediolani, 1708 (168 pages) (discussing questions of statics and dynamics).

His third mathematical work *Euclides ab omni nævo vindicatus* was published at Milan in 1733 (16 + 142 pages + 6 plates). Saccheri died in October, 1733, and there is doubt as to whether he lived to see his completed master work issue from the press. While the work is frequently referred to during the next one hundred and fifty years it was not till the publication of an article by Beltrami[1] in 1889 that Saccheri became generally recognized as a forerunner of Legendre, Lobatchevsky, and Bolyai. Indeed Saccheri gave many of their propositions. For example, in his discussions of the parallel postulate, 1794–1833, Legendre proved, by using only the first twenty-eight propositions of Euclid's Elements, that: The sum of the angles of a triangle cannot be greater than two right angles; and that the sum must be equal to two right angles if this is true for a single triangle. Both of these propositions are proved in more general form by Saccheri.

In Saccheri's time the conception of parallels as equidistant straight lines was a favorite one, but Saccheri, like some of his predecessors, as Sommerville remarks, "saw that it would not do to assume this in the definition. He starts with two equal perpendiculars AC and BD to a line AB. When the ends C, D

[1] "Un precursore italiano di Legendre et di Lobatschewsky," *Atti della Reale Accademia dei Lincei*, Anno 1889, series 4, Vol. 5, pp. 441–448.